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Symmetries of Lagrangians and of their equations of motion

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Abstract. A new kind of Lagrangian symmetry is defined in such a way that the resulting set of Lagrangian symmetries coincides with the set of symmetries of its equations of motion. Several constants of motion may be associated to each of the new symmetry transformations. One example is presented.

1. Introduction

The subject of symmetries and associated constants of motion plays a prominent role in physics. If a physical system may be described in terms of a Lagrangian it is customary to study the symmetries of the Lagrangian to get information about (at least some of) the constants of motion associated to the dynamical problem. One of the most useful tools to explore Lagrangian symmetries and construct associated constants of motion is the very well known theorem due to Noether (Noether 1918, Hill 1951, Lovelock and Rund 1975, Sudarshan and Mukunda 1974) which, in addition to providing a simple way to construct a constant of motion for each symmetry transformation, constitutes the starting point for building gauge theories (Utiyama 1956, Abers and Lee 1973) which so successfully describe fundamental interactions.

Much discussion has been devoted to the subject (see e.g. Havas 1973) and there are still some points which seem to remain unclear. This article is devoted to one of them, namely, the relationship between the symmetries of Lagrangians and those of their equations of motion and the construction of the associated constants of motion.

Even though the discussion about this subject is meaningless unless the concept of a symmetry transformation is clearly defined, it can be safely said, before going into details, that the general belief seems to be that Lagrangians do, in general, possess less symmetry than their associated equations of motion. This is due to the fact that one is usually defining a Lagrangian symmetry to be one which satisfies the assumptions made in Noether's theorem.

Recently (Hojman 1980, Hojman and Harleston 1980, 1981, Hojman and Urrutia 1981, Hojman and Shepley 1982) a new concept of Lagrangian symmetry, called s-equivalence, has been defined, generalising the one due to Noether, in such a way that *several* constants of motion may be associated to *one* symmetry transformation (Hojman and Gómez 1984). If s-equivalence is taken into account, the set of symmetries of Lagrangians and their equations of motion coincide, if the latter are first-order differential equations (Hojman and Zertuche 1984).

In this article, we study the case of second-order differential equations. In this case, s-equivalence is not sufficient and a new kind of Lagrangian symmetry has to be

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defined so that the set of symmetries of a Lagrangian and that of its equations of motion are the same. This new kind of symmetry can also be related to several constants of motion. This Lagrangian symmetry, which seems to have remained unnoticed up to now, allows one to state that the symmetries of a Lagrangian and those of its equations of motion are the same.

In § 2, a review of the different definitions of symmetry transformations for Lagrangians and equations of motion is presented. In § 3, recent results on the inverse problem of the calculus of variations, relevant for the purposes of this article, are summarised. In § 4, the new symmetry is defined and is related to constants of motion. This new definition is constructed so that the symmetries of Lagrangians and their equations of motion are the same. In § 5, one example is presented. Section 6 contains conclusions.

Detailed calculations leading to some equations (marked with an asterisk in the main text) are given in an appendix.

2. Definitions of symmetry transformations for Lagrangians and equations of motion

Consider a quasilinear second-order differential system of equations

$$M_{ij}(q^k, \dot{q}^k, t)\ddot{q}^j + N_i(q^k, \dot{q}^k, t) = 0, \qquad i, j, k = 1, \ldots, n, \tag{2.1}$$

and assume further that M_{ij} is regular, i.e.

$$\det(M_{ij}) \neq 0, \tag{2.2}$$

and therefore (2.1) is equivalent to

$$\ddot{q}^i - F^i(q^j, \dot{q}^j, t) = 0 \tag{2.3}$$

where

$$F^i = -(M^{-1})^{ij} N_j. \tag{2.4}$$

Consider an infinitesimal transformation of coordinates

$$q'^i = q^i + \varepsilon\eta^i(q^j, \dot{q}^j, t). \tag{2.5}$$

The change of coordinates defined by (2.5) is said to be a symmetry transformation for (2.1) (or (2.3)) if it maps any solution of (2.1) into another solution of (2.1), i.e. if η^i satisfies (2.6) to within terms of order ε^2,

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\eta^i - \frac{\partial F^i}{\partial \dot{q}^j}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\eta^j - \frac{\partial F^i}{\partial q^j}\eta^j = 0 \tag{2.6}$$

where

$$\bar{\mathrm{d}}/\mathrm{d}t \equiv F^i\,\partial/\partial\dot{q}^i + \dot{q}^i\,\partial/\partial q^i + \partial/\partial t. \tag{2.7}$$

(Santilli 1978) equation (2.6) is sometimes called the equation of variations of (2.3) and its solutions are transformations of symmetry for (2.1) and (2.3).

It is perhaps worthwhile mentioning that if $C(q^i, \dot{q}^i, t)$ is a constant of motion for a given problem and $\eta^i(q^j, \dot{q}^j, t)$ satisfies (2.6) then $C'(q^i, \dot{q}^i, t)$,

$$C' = (\partial C/\partial q^i)\eta^i + (\partial C/\partial \dot{q}^i)\,\bar{\mathrm{d}}\eta^i/\mathrm{d}t,$$

is also a constant of motion (Katzin and Levine 1975).

Let us now turn our attention to Lagrangian systems and Noether's theorem. Consider the Lagrangian

$$L = L(q^i, \dot{q}^i, t) \tag{2.8}$$

and its Euler–Lagrange equations

$$E_i L = 0 \tag{2.9}$$

where

$$E_i L \equiv (\mathrm{d}/\mathrm{d}t)\, \partial L/\partial \dot{q}^i - \partial L/\partial q^i. \tag{2.10}$$

Define the infinitesimal change of coordinates and time

$$q'^i = q^i - \delta q^i(q^j, \dot{q}^j, t), \tag{2.11}$$

$$t' = t - \delta t(q^j, \dot{q}^j, t). \tag{2.12}$$

It is convenient to define

$$\delta L = \frac{\partial L}{\partial q^i}\,\delta q^i + \frac{\partial L}{\partial \dot{q}^i}\,(\delta q^i)^{\cdot} + \frac{\partial L}{\partial t}\,\delta t + \left(L - \frac{\partial L}{\partial \dot{q}^i}\,\dot{q}^i \right)(\delta t)^{\cdot}. \tag{2.13}$$

Noether's theorem asserts that if δL can be written as a total time derivative, i.e.

$$\delta L \equiv -\mathrm{d}f(q^i, \dot{q}^i, t)/\mathrm{d}t, \tag{2.14}$$

then K is a conserved quantity

$$K = (\partial L/\partial \dot{q}^i)\delta q^i + [L - (\partial L/\partial \dot{q}^i)\dot{q}^i]\delta t + f. \tag{2.15}$$

The proof is straightforward using definition (2.15) and (2.9), (2.13) and (2.14).

If δq^i and δt satisfy (2.14) it is usually said that (2.10) is a (Noetherian) symmetry transformation for Lagrangian (2.8).

In what follows we will briefly restate condition (2.14) in a slightly different (but equivalent) way.

Define the (acceleration dependent) Euler–Lagrange operator G_i

$$G_i \equiv -(\mathrm{d}^2/\mathrm{d}t^2)\, \partial/\partial \ddot{q}^i + E_i \tag{2.16}$$

where E_i is defined by (2.9), which will be very useful in § 3. Of course, (2.9) can be rewritten as

$$G_i L = 0. \tag{2.17}$$

The (Noetherian) symmetry requirement (2.14) can then be re-expressed in words by saying that the Euler–Lagrange derivatives of δL vanish identically, i.e.

$$G_i \delta L \equiv 0. \qquad (2.18)^*$$

Noether's theorem then relates one conserved quantity K defined by (2.15) to each (Noetherian) symmetry transformation defined by (2.10) and (2.14) (or (2.18)).

Consider now a more general situation. Assume that the Euler–Lagrange derivatives of δL do not vanish identically but are a linear combination of the Euler–Lagrange derivatives of L

$$G_i \delta L = \Lambda_i{}^j(q^k, \dot{q}^k, t) G_j L. \tag{2.19}$$

In this case, the Euler–Lagrange derivatives of δL vanish only when the equations of motion (2.9) or (2.17) are satisfied,

$$G_i \delta L|_{G_i L=0} = 0 \tag{2.20}$$

i.e. they vanish on the space of solutions of (2.9) or (2.17). If det $\Lambda = 0$, δL is subordinated to L (Currie and Saletan 1966). If det $\Lambda \neq 0$, then δL and L are s-equivalent, i.e. their equations of motion have exactly the same solutions (Hojman and Harleston 1981). The transformations defined by (2.11) and (2.12) satisfying requirement (2.19) are called non-Noetherian or s-equivalence symmetry transformations.

The conservation laws associated to this symmetry transformation are

$$\operatorname{Tr} \Lambda^k = \text{constant}, \qquad k = 1, 2, \ldots, \tag{2.21}$$

(Hojman 1980, Hojman and Harleston 1980, 1981, Hojman and Urrutia 1981, Hojman and Gómez 1984). Non-Noetherian symmetries associate *several* conserved quantities to *one* symmetry transformation (at most n of them are functionally independent).

It is interesting to compare Noetherian and non-Noetherian symmetries of a given Lagrangian with those of its equations of motion. For simplicity, we take $\delta t = 0$ for the time being, but in § 4 the general case will also be discussed. It can be proved (see § 3) that Noetherian and non-Noetherian symmetries satisfy (2.6), where F^i is defined by

$$F^i = -(W^{-1})^{ij} V_j \tag{2.22}$$

with

$$W_{ij} \equiv \partial^2 L / \partial \dot{q}^i \, \partial \dot{q}^j = W_{ji}, \tag{2.23}$$

$$V_i \equiv (\partial^2 L / \partial q^j \partial \dot{q}^i) \dot{q}^j + \partial^2 L / \partial t \, \partial \dot{q}^i - \partial L / \partial q^i \tag{2.24}$$

(we assume throughout this work that

$$\det W \neq 0 \tag{2.25}$$

and therefore W^{-1} exists).

Conversely, it is easy to see that *not* all solutions to (2.6) are either Noetherian or non-Noetherian symmetry transformationns.

A new kind of (Lagrangian) symmetry transformation has to be defined in order to have equivalence between symmetries of a given Lagrangian and those of its equations of motion. We review in § 3 very recent results on the inverse problem of the calculus of variations which will be very useful to achieve this goal and construct conserved quantities associated to this new kind of symmetry.

3. Lagrangians as linear combinations of their own Euler–Lagrange derivatives

In this section we briefly review some of the results obtained recently in the inverse problem of the calculus of variations. This problem consists in studying the existence and uniqueness (or multiplicity) of Lagrangians for a given system of differential equations (or any other system which has the same general solution).

Consider the equations

$$\ddot{q}^i - F^i(q^j, \dot{q}^j, t) = 0. \tag{3.1}$$

It has been shown (Hojman *et al* 1983) that any Lagrangian for (3.1) may be written as a linear combination of (the left-hand side of) equations (3.1) themselves

$$\tilde{L} = \tilde{L}(q^i, \dot{q}^i, \ddot{q}^i, t) = \mu_i(q^j, \dot{q}^j, t)(\ddot{q}^i - F^i) \tag{3.2}$$

up to a total time derivative.

Lagrangian (3.2) is unusual in the sense that it is acceleration dependent but as will be seen shortly, it differs from a usual (acceleration independent) Lagrangian by a total time derivative.

The coefficients μ_i have to satisfy

$$\gamma_{ij} = 0, \tag{3.3}$$

$$(\bar{\mathrm{d}}/\mathrm{d}t)(\bar{\mathrm{d}}\mu_i/\mathrm{d}t + \mu_k \partial F^k/\partial \dot{q}^i) - \mu_k\, \partial F^k/\partial q^i = 0, \tag{3.4}$$

$$\det \alpha_{ij} \neq 0, \tag{3.5}$$

where

$$\gamma_{ij} \equiv \partial\mu_i/\partial\dot{q}^j - \partial\mu_j/\partial\dot{q}^i = -\gamma_{ji}, \tag{3.6}$$

$$\alpha_{ij} = (\partial/\partial\dot{q}^j)(\bar{\mathrm{d}}\mu_i/\mathrm{d}t + \mu_k\, \partial F^k/\partial\dot{q}^i) + \partial\mu_j/\partial q^i, \tag{3.7}$$

and

$$\bar{\mathrm{d}}/\mathrm{d}t \equiv F^i\, \partial/\partial\dot{q}^i + \dot{q}^i\, \partial/\partial q^i + \partial/\partial t, \tag{3.8}$$

in order that

$$G_i\tilde{L} = 0 \tag{3.9}$$

be equivalent to (3.1).

Condition (3.3) guarantees that no third-order derivatives appear in $G_i\tilde{L}$ (there are no fourth-order derivatives anyway because L is linear in the accelerations). Condition (3.4) implies that

$$G_i\tilde{L} = -\alpha_{ij}(\ddot{q}^j - F^j) \tag{3.10}$$

and therefore (3.9) are satisfied once (3.1) are. Condition (3.5) ensures that the converse is true, i.e. that (3.9) imply (3.1).

Condition (3.3) can also be understood in a different way. In fact, (3.3) and (3.6) imply that a function $g(q^i, \dot{q}^i, t)$ exists such that

$$\mu_i = -\partial g/\partial\dot{q}^i \tag{3.11}$$

and therefore the Lagrangian L,

$$L(q^i, \dot{q}^i, t) = \tilde{L}(q^i, \dot{q}^i, \ddot{q}^i, t) + \mathrm{d}g/\mathrm{d}t, \tag{3.12}$$

is acceleration independent. As a matter of fact,

$$L(q^i, \dot{q}^i, t) = (\bar{\mathrm{d}}/\mathrm{d}t)g(q^i, \dot{q}^i, t). \tag{3.13}$$

Of course,

$$G_iL \equiv E_iL \equiv G_i\tilde{L}. \tag{3.14}$$

So the equations of motion for L can be written in the usual way. If one has the Lagrangian given in the form of (3.2) there is no need to compute g to get the equations of motion which can be written in the form (3.9) as long as conditions (3.3)–(3.5) are satisfied.

Integrability conditions for (3.3) and (3.4) are

$$\bar{\mathrm{d}}\alpha_{ij}/\mathrm{d}t = -\tfrac{1}{2}[\alpha_{ik}\,\partial F^k/\partial \dot{q}^j + (\partial F^k/\partial \dot{q}^i)\alpha_{kj}], \tag{3.15}$$

$$\bar{\mathrm{d}}\beta_{ij}/\mathrm{d}t = \alpha_{ik}\,\partial F^k/\partial q^j - (\partial F^k/\partial q^i)\alpha_{kj}, \tag{3.16}$$

$$\beta_{ij} = -\tfrac{1}{2}[(\partial F^k/\partial \dot{q}^i)\alpha_{kj} - \alpha_{ik}\,\partial F^k/\partial \dot{q}^j], \tag{3.17}$$

where β_{ij} is defined by

$$\beta_{ij} \equiv (\partial/\partial q^i)(\bar{\mathrm{d}}\mu_j/\mathrm{d}t + \mu_k\,\partial F^k/\partial \dot{q}^j) - (i \leftrightarrow j). \tag{3.18}$$

Furthermore, if (3.3) is satisfied α_{ij} is such that

$$\alpha_{ij} = \alpha_{ji} \tag{3.19}$$

and

$$\partial \alpha_{ij}/\partial \dot{q}^k = \partial \alpha_{ik}/\partial \dot{q}^j. \tag{3.20}$$

Equations (3.15)–(3.20) are the starting point in the approaches devised by Henneaux (1982) and Sarlet (1982). They have obtained further conditions from (3.15)–(3.20) which will not be needed in what follows.

Define, for an acceleration independent Lagrangian $L(q^i, \dot{q}^i, t)$, W_{ij} and T_{ij} by

$$W_{ij} \equiv \partial^2 L/\partial \dot{q}^i\,\partial \dot{q}^j = W_{ji}, \tag{3.21}$$

$$T_{ij} = \partial^2 L/\partial \dot{q}^i\,\partial q^j - \partial^2 L/\partial \dot{q}^j\,\partial q^i = -T_{ji}. \tag{3.22}$$

It is straightforward to prove, using (3.3), (3.4) and (3.13), that

$$\alpha_{ij} = -W_{ij}, \tag{3.23}$$

$$\beta_{ij} = T_{ij} \tag{3.24}$$

(for such an acceleration independent Lagrangian γ_{ij}, of course, vanishes).

Suppose now that a symmetric, non-singular matrix S_{ij} satisfying the conditions (3.15), (3.19) and (3.20) that α_{ij} has to fulfil for a given F^k is known. It can be proved (Hojman *et al* 1983) that if θ^j satisfies (2.6) then ρ_i,

$$\rho_i = S_{ij}\theta^j, \tag{3.25}$$

is a solution of (3.4) and conversely, if ρ_i fulfils (3.4) then θ^j satisfies (2.6). Of course,

$$S_{ij} = W_{ij} \tag{3.26}$$

is a possible choice if W_{ij} is defined in terms of a Lagrangian for which (2.22) is satisfied. If an acceleration dependent Lagrangian is used, then the choice

$$S_{ij} = \alpha_{ij} \tag{3.27}$$

is also a good one.

The function $\tilde{L}'$ defined by

$$\tilde{L}' = \theta^i S_{ij}(\ddot{q}^j - F^j) \tag{3.28}$$

will be a new Lagrangian subordinated (or s-equivalent) to the original one only if condition (3.3) is met, i.e. if

$$S_{ik}\,\partial \theta^k/\partial \dot{q}^j - (\partial \theta^k/\partial \dot{q}^i)S_{jk} = 0. \tag{3.29}$$

4. A new Lagrangian symmetry

Consider the variation of the Lagrangian δL given by

$$\delta L = (\partial L/\partial q^i)\delta q^i + (\partial L/\partial \dot{q}^i)(\delta q^i)^{\cdot} \qquad (\delta t = 0). \tag{4.1}$$

It is very well known that δL can be re-expressed in the following way,

$$\delta L = ((\partial L/\partial \dot{q}^i)\delta q^i)^{\cdot} - \delta q^i W_{ij}(\ddot{q}^j - F^j), \tag{4.2*}$$

as can be easily checked (Sudarshan and Mukunda 1974). The matrix W_{ij} is defined by (3.21). Therefore, δL can be written (up to a total time derivative) as a linear combination of the (left-hand side of the) equations of motion of L. The matrix W_{ij} satisfies (3.15), (3.19) and (3.20) by construction as was already mentioned at the end of § 3. Therefore, $\delta q^i W_{ij}$ fulfils condition (3.4) if and only if δq^i is a solution of the equation of variations (2.6). Furthermore, if δq^i satisfies

$$W_{ik}\, \partial \delta q^k/\partial \dot{q}^j - (\partial \delta q^k/\partial \dot{q}^i)\, W_{kj} = 0 \tag{4.3}$$

δL is either subordinate to L (including the possibility of a total time derivative) or s-equivalent to L.

Consider now the case in which δq^i does not satisfy (4.3). What happens then? Compute

$$G_i \delta L = C_{ij}(\ddot{q}^j - F^j)^{\cdot} + (-A_{ij} + \dot{C}_{ij})(\ddot{q}^j - F^j) \tag{4.4}$$

where

$$A_{ij} = \frac{\partial}{\partial \dot{q}^j}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}(W_{ik}\delta q^k) + W_{kl}\delta q^l \frac{\partial F^k}{\partial \dot{q}^i}\right) + \frac{\partial (W_{jk}\delta q^k)}{\partial q^i}, \tag{4.5}$$

$$C_{ij} = \partial(W_{jk}\delta q^k)/\partial \dot{q}^i - \partial(W_{ik}\delta q^k)/\partial \dot{q}^j, \tag{4.6}$$

i.e. the Euler–Lagrange derivatives of δL are linear combinations of the left-hand side of the equations of motion of L and their time derivatives. In other words, the Euler–Lagrange derivatives of δL vanish when evaluated in the space of solutions of the equations of motion of L or

$$G_i \delta L|_{G_i L = 0} = 0. \tag{4.7}$$

Condition (4.4) (or (4.7)) can be used to define a symmetry transformation for a given Lagrangian and includes Noether and s-equivalence symmetries, when (4.3) is fulfilled ($C_{ij} = 0$), and a new kind of symmetry when C_{ij} does not vanish. Condition (4.4) (or (4.7)) is such that taking it as a definition of Lagrangian symmetry, it makes symmetries of Lagrangians and those of equations of motion become equivalent.

When C_{ij} vanishes the conservation laws are known and appear in § 2. In what follows, we will find the conservation law associated to the general case when C_{ij} does not vanish. For this purpose it is useful to define l_a $(a = 1, 2, \ldots, 2n)$,

$$l_{i+n} = W_{ij}\delta q^j, \tag{4.8}$$

$$l_i = -(\bar{\mathrm{d}} l_{i+n}/\mathrm{d}t + l_{k+n}\, \partial F^k/\partial \dot{q}^i). \tag{4.9}$$

It is straightforward to prove that if l_{i+n} satisfies (3.4) then l_a is such that

$$\bar{\mathrm{d}} l_a/\mathrm{d}t + l_b\, \partial f^b/\partial x^a = 0, \qquad a, b = 1, 2, \ldots, 2n, \tag{4.10}$$

where

$$x^i = q^i, \qquad x^{i+n} = \dot{q}^i, \qquad f^i = x^{i+n}, \qquad f^{i+n} = F^i(x^j, x^{j+n}, t), \tag{4.11}$$

$$\bar{\mathrm{d}}/\mathrm{d}t = f^a\, \partial/\partial x^a + \partial/\partial t. \tag{4.12}$$

Define

$$\sigma_{ab} = \partial l_a/\partial x^b - \partial l_b/\partial x^a = -\sigma_{ba} \tag{4.13}$$

which satisfies

$$\bar{\mathrm{d}}\sigma_{ab}/\mathrm{d}t = \sigma_{bc}\, \partial f^c/\partial x^a - \sigma_{ac}\, \partial f^c/\partial x^b \tag{4.14*}$$

because of (4.10). As a matter of fact σ_{ab} can also be written as

$$\sigma = \begin{pmatrix} -B & -A \\ A^{\mathrm{T}} & -C \end{pmatrix}, \qquad A^{\mathrm{T}}_{ij} \equiv A_{ji}, \tag{4.15}$$

where A and C are defined by (4.5) and (4.6) respectively and

$$B_{ij} = -(\partial l_i/\partial q^j - \partial l_j/\partial q^i). \tag{4.16}$$

The regular $2n \times 2n$ matrix H_{ab} $(= -H_{ba})$,

$$H = \begin{pmatrix} T & W \\ -W & 0 \end{pmatrix}, \tag{4.17*}$$

with W and T defined by (3.21) and (3.22) also satisfies (4.14). Therefore,

$$\mathrm{Tr}\ V^k = \text{constant}, \qquad k = 1, 2, \ldots, \tag{4.18}$$

where

$$V = \sigma H^{-1} \tag{4.19}$$

because σ and H satisfy (4.14) (see Hojman and Urrutia 1981).

We have hence found several conservation laws associated to the new kind of Lagrangian symmetry defined, related to the traces of powers of a $2n \times 2n$ matrix. When C vanishes, the traces of the powers of V are twice (see appendix) those of the corresponding powers of Λ,

$$\Lambda = -AW^{-1}. \tag{4.20}$$

Consider now $\delta t \neq 0$. Equation (2.6) is modified but may still be written in the same way for a variable $\bar{\eta}^i$ defined by

$$\bar{\eta}^i = \eta^i - \dot{q}^i \delta t. \tag{4.21*}$$

The variation of the Lagrangian δL is

$$\delta L = \left(\frac{\partial L}{\partial \dot{q}^i}\, \delta q^i + \left(L - \frac{\partial L}{\partial \dot{q}^i}\, \dot{q}^i \right) \delta t \right) - (\delta q^i - \dot{q}^i \delta t)\, W_{ij} (\ddot{q}^j - F^j) \tag{4.22*}$$

instead of expression (4.2). One can define $\bar{\delta} q^i$ by

$$\bar{\delta} q^i = \delta q^i - \dot{q}^i \delta t \tag{4.23}$$

and everything can now be restated in terms of $\bar{\eta}^i$ and $\bar{\delta} q^i$, and the same results hold.

It is straightforward to prove that

$$J = l_a \xi^a, \qquad a = 1, 2, \ldots, 2n, \tag{4.24}$$

is also a constant of motion where

$$\xi^i = \eta^i, \qquad \xi^{i+n} = \bar{d}\eta^i/dt \tag{4.25}$$

using (4.10), (4.11), (2.6) and (4.25) (because ξ^a satisfies

$$\bar{d}\xi^a/dt - \xi^b \partial f^a/\partial x^b = 0). \tag{4.26}$$

One last remark. If instead of starting from a usual (acceleration independent) Lagrangian $L(q^i, \dot{q}^i, t)$ one is given an acceleration dependent Lagrangian $\tilde{L}(q^i, \dot{q}^i, \ddot{q}^i, t)$ differing from L by a total time derivative, its variation $\delta\tilde{L}$ will differ from δL by a total time derivative and the conclusions remain unchanged.

5. Example

The example is devoted to illustrating the third kind of Lagrangian symmetry defined in this paper (examples for Noetherian and s-equivalence symmetry transformation can be found in Hill (1951), Hojman and Harleston (1981) and Hojman and Gómez (1984)).

Consider the Lagrangian for the two-dimensional harmonic oscillator

$$L = \tfrac{1}{2} \sum_{i=1}^{2} (\dot{q}_i^2 - q_i^2). \tag{5.1}$$

The equations of motion are

$$\ddot{q}_i + q_i = 0, \qquad i = 1, 2. \tag{5.2}$$

The equation of variations is

$$(\bar{d}/dt)(\bar{d}/dt)\eta^i + \eta^i = 0. \tag{5.3}$$

A solution to (5.3) is

$$\eta^i = Eq^i \tag*{(5.4)*}$$

where E is the conserved energy

$$E = \tfrac{1}{2} \sum_{i=1}^{2} (\dot{q}_i^2 + q_i^2). \tag{5.5}$$

The variation of the Lagrangian δL can be readily constructed,

$$\delta L = \sum_{i=1}^{2} \left[-Eq_i^2 + \dot{q}_i \left(E\dot{q}_i + \sum_{j=1}^{2} (\dot{q}_j \ddot{q}_j + q_j \dot{q}_j) q_i \right) \right]. \tag*{(5.6)*}$$

The Euler–Lagrange derivatives of δL are

$$G_i \delta L = C_{ij}(\dddot{q}_j + \dot{q}_j) + (-A_{ij} + \dot{C}_{ij})(\ddot{q}_j + q_j) \tag*{(5.7)*}$$

where

$$A_{ij} = 2E\delta_{ij} + q_i q_j + \dot{q}_i \dot{q}_j, \tag{5.8}$$

$$C_{ij} = \dot{q}_i q_j - \dot{q}_j q_i. \tag{5.9}$$

Note that $G_i \delta L$ vanishes if both (5.2) and its time derivative hold.

Furthermore, B can be constructed,

$$B_{ij} = \dot{q}_i q_j - \dot{q}_j q_i, \tag{5.10}$$

and therefore σ is

$$\sigma = \begin{pmatrix} \dot{q}_i q_j - \dot{q}_j q_i & 2E\delta_{ij} + q_i q_j + \dot{q}_i \dot{q}_j \\ -(2E\delta_{ij} + q_i q_j + \dot{q}_i \dot{q}_j) & \dot{q}_i q_j - \dot{q}_j q_i \end{pmatrix}. \tag{5.11}$$

Let us now turn our attention to L. The matrices W and T are

$$W_{ij} = \delta_{ij}, \tag{5.12}$$

$$T_{ij} = 0. \tag{5.13}$$

The matrix H is

$$H = \begin{pmatrix} 0 & \delta_{ij} \\ -\delta_{ij} & 0 \end{pmatrix} \tag{5.14}$$

and therefore V can be written as

$$V = \begin{pmatrix} 2E\delta_{ij} + q_i q_j + \dot{q}_i \dot{q}_j & -\dot{q}_i q_j + q_j q_i \\ \dot{q}_i q_j - \dot{q}_j q_i & 2E\delta_{ij} + q_i q_j + \dot{q}_i \dot{q}_j \end{pmatrix}. \tag{5.15}$$

It is straightforward to check that the traces of all powers of V are constants of motion,

$$\operatorname{Tr} V^k = \text{constant}, \qquad k = 1, 2, \ldots . \tag{5.16}$$

We have then showed that transformation (5.4) which is a solution to (2.6) is neither a Noetherian symmetry nor an s-equivalence one and therefore belongs to the third kind of Lagrangian symmetries defined in this paper. Constants of motion (5.16) can be associated to it.

6. Conclusions and outlook

We have introduced a new definition of Lagrangian symmetry in such a way that the sets of symmetry transformations of Lagrangians and their equations of motion coincide. Conservation laws can be associated to this newly defined Lagrangian symmetry much in the same way as Noetherian and s-equivalence symmetry transformations.

The constant of motion $C'(q^i, \dot{q}^i, t)$,

$$C' = (\partial C/\partial q^i)\eta^i + (\partial C/\partial \dot{q}^i)\,\bar{\mathrm{d}}\eta^i/\mathrm{d}t,$$

where C is a constant of motion for a given problem, and $\eta^i(q^j, \dot{q}^j, t)$ satisfies (2.6) for the same problem, constitutes a generalisation of Poisson's theorem for constants of motion.

The new concept of Lagrangian symmetry may be used in the construction of gauge theories which are usually built considering Noetherian symmetries only.

The conservation laws obtained may be used to integrate nonlinear systems much in the same way as the Lax method (Lax 1968) is used. This procedure will have the advantage of having a way of linking symmetries and the conserved traces of powers of matrices.

Finally, it would be interesting to compare the methods and results of this work with those of Crampin (1983), Prince (1983) and Sarlet (1983).

Appendix

Equation (2.14) is equivalent to (2.18)

If (2.14) holds, it is straightforward to prove that (2.18) is satisfied. The converse is not so obvious. Assume that

$$G_i\Delta(q^j, \dot{q}^j, \ddot{q}^j, t) \equiv 0. \tag{A1}$$

To prove that

$$\Delta = -(\mathrm{d}/\mathrm{d}t)f(q^i, \dot{q}^i, t) \tag{A2}$$

we proceed by inspecting $G_i\Delta$. Because (A1) is an identity, the coefficients of the highest derivatives of q^i in $G_i\Delta$ must vanish identically. The highest derivatives of q^i in $G_i\Delta$ are $\ddddot{q}^i$ and the vanishing of its coefficients implies that Δ is linear in $\ddot{q}^i$, i.e.

$$\partial^2\Delta/\partial\ddot{q}^i\,\partial\ddot{q}^j = 0 \tag{A3}$$

or

$$\Delta = \Delta_i(q^j, \dot{q}^j, t)\ddot{q}^i + \Delta_0(q^j, \dot{q}^j, t). \tag{A4}$$

Now, the highest derivative of q^i left in $G_i\Delta$ is $\dddot{q}^i$, and its coefficient must vanish, i.e.

$$\partial\Delta_i/\partial\dot{q}^j - \partial\Delta_j/\partial\dot{q}^i = 0 \tag{A5}$$

or

$$\Delta_i = \partial g(q, \dot{q}, t)/\partial\dot{q}^i.$$

Now, vanishing of the coefficient of $\ddot{q}^k$ in $G_i\Delta$ implies

$$\partial h_k/\partial\dot{q}^i = 0, \qquad \text{i.e. } h_k = h_k(q, t) \tag{A6}$$

where

$$h_k \equiv \left(\frac{\partial g}{\partial q^k} + \frac{\partial^2 g}{\partial\dot{q}^k\,\partial q^j}\dot{q}^j + \frac{\partial^2 g}{\partial\dot{q}^k\,\partial t} - \frac{\partial\Delta_0}{\partial\dot{q}^k}\right). \tag{A7}$$

Note that

$$h_k(q, t) = \partial\rho/\partial\dot{q}^k \tag{A8}$$

where

$$\rho \equiv -\Delta_0 + (\partial g/\partial q^j)\dot{q}^j + \partial g/\partial t; \tag{A9}$$

therefore,

$$\rho \equiv h_k(q, t)\dot{q}^k + h_0(q, t). \tag{A10}$$

Now vanishing of $G_i\Delta$ implies

$$\partial\rho/\partial q^i \equiv \mathrm{d}h_i/\mathrm{d}t, \tag{A11}$$

i.e.

$$\partial h_l/\partial q^i = \partial h_i/\partial q^l \qquad \text{and} \qquad \partial h_0/\partial q^i = \partial h_i/\partial t, \tag{A12}$$

or

$$h_l = \partial h(q, t)/\partial q^l, \qquad h_0 = \partial h(q, t)/\partial t, \tag{A13}$$

$$\rho = \frac{\mathrm{d}h}{\mathrm{d}t} \qquad \text{or} \qquad \Delta_0 = \frac{\partial g}{\partial q^j}\dot{q}^j + \frac{\partial g}{\partial t} + \frac{\mathrm{d}h}{\mathrm{d}t}. \tag{A14}$$

Therefore

$$\Delta = \Delta_i \ddot{q}^i + \Delta_0 = (\mathrm{d}/\mathrm{d}t)(g+h) \equiv -\mathrm{d}f/\mathrm{d}t, \tag{A15}$$

i.e. (2.18) implies (2.14).

Proof of (4.14)

To prove (4.14) it is convenient to consider the identity

$$(\partial/\partial x^a)\bar{\mathrm{d}}/\mathrm{d}t = (\bar{\mathrm{d}}/\mathrm{d}t)\,\partial/\partial x^a + f^b_{,a}\,\partial/\partial x^b. \tag{A16}$$

Now, by definition (4.13)

$$\bar{\mathrm{d}}\sigma_{ab}/\mathrm{d}t = (\bar{\mathrm{d}}/\mathrm{d}t)(\partial l_a/\partial x^b - \partial l_b/\partial x^a) \tag{A17}$$

or, using identity (A16),

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\sigma_{ab} = \frac{\partial}{\partial x^b}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}l_a\right) - f^c_{,b}\frac{\partial l_a}{\partial x^c} - \frac{\partial}{\partial x^a}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}l_b\right) + f^c_{,a}\frac{\partial l_b}{\partial x^c}. \tag{A18}$$

Now, with the help of (4.10) and definition (4.13) one gets (4.14).

Proof that H defined by (4.17) satisfies (4.14)

The proof is straightforward when definition (4.17) is considered together with (3.15), (3.16), (3.17), (3.23), (3.24) and (4.11).

If C vanishes, then Tr $V^k = 2 Tr\, \Lambda^k$

$$\sigma = \begin{pmatrix} -B & -A \\ A & 0 \end{pmatrix} \tag{A19}$$

and V is

$$V = \begin{pmatrix} -AW^{-1} & Y_1 \\ 0 & -AW^{-1} \end{pmatrix}. \tag{A20}$$

It is straightforward to prove that

$$V^k = \begin{pmatrix} \Lambda^k & Y_k \\ 0 & \Lambda^k \end{pmatrix} \tag{A21}$$

so that

$$\mathrm{Tr}\, V^k = 2\mathrm{Tr}\, \Lambda^k. \tag{A22}$$

The explicit form of the matrices Y_k is irrelevant for this proof.

Why (4.21) works for (2.6)

Consider

$$q'^i = q^i + \varepsilon\eta^i(q^j, \dot{q}^j, t), \qquad t' = t + \varepsilon\eta^0(q^j, \dot{q}^j, t), \tag{A23a, b}$$

instead of transformation (2.5). Then one gets

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\eta^i - 2\ddot{q}^i\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\eta^0 - \dot{q}^i\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\eta^0 - \frac{\partial F^i}{\partial \dot{q}^j}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\eta^i - \dot{q}^j\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\eta^0\right) - \frac{\partial F^i}{\partial q^j}\eta^j - \frac{\partial F^i}{\partial t}\eta^0 = 0 \tag{A24}$$

instead of (2.6), because of

$$\bar{\mathrm{d}}/\mathrm{d}t' = (1 - \bar{\mathrm{d}}\eta^0/\mathrm{d}t)\,\bar{\mathrm{d}}/\mathrm{d}t. \tag{A25}$$

It is straightforward to see that (A24) is equivalent to (2.6) written for $\bar{\eta}^i$ defined by (4.21) with $\delta t = \eta^0$.

Derivation of (5.4), (5.6) and (5.7)

Consider (2.6); it is straightforward to prove that if η^i is a solution of (2.6), then $K\eta^i$ is also a solution of (2.6) if K is a constant of motion, i.e. if

$$\bar{\mathrm{d}}K/\mathrm{d}t = 0. \tag{A26}$$

For (5.3), $\eta_0^i = q^i$ is a solution. Therefore $\eta^i = Eq^i$ is also a solution. Equations (5.6) and (5.7) can be readily obtained from definitions (2.13) and (2.16) respectively.

Detailed derivation of (4.22) and (4.2)

Consider the identity

$$\frac{\partial L}{\partial t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(L - \frac{\partial L}{\partial \dot{q}^i}\dot{q}^i\right) - \left(\frac{\partial L}{\partial q^i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^i}\right)\dot{q}^i. \tag{A27}$$

Add and subtract $(\mathrm{d}/\mathrm{d}t)(\partial L/\partial \dot{q}^i)\delta q^i$ to the definition (2.13) for δL, use expressions (2.22), (2.23) and (2.24) for $(\mathrm{d}/\mathrm{d}t)(\partial L/\partial \dot{q}^i) - \partial L/\partial q^i$ and identity (A27) to get (4.22) for δL. Take $\delta t = 0$ and get (4.2).

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